

Explicit solutions of quadratic FBSDEs arising from quadratic term structure models

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Abstract

We provide explicit solutions of certain forward-backward stochastic differential equations (FBSDEs) with quadratic growth. These particular FBSDEs are associated with quadratic term structure models of interest rates and characterize the zero-coupon bond price. The results of this paper are naturally related to similar results on affine term structure models of Hyndman (Math. Financ. Econ. 2(2):107-128, 2009) due to the relationship between quadratic functionals of Gaussian processes and linear functionals of affine processes. Similar to the affine case a sufficient condition for the explicit solutions to hold is the solvability in a fixed interval of Riccati-type ordinary differential equations. However, in contrast to the affine case, these Riccati equations are easily associated with those occurring in linear-quadratic control problems. We also consider quadratic models for a risky asset price and characterize the futures price and forward price of the asset in terms of similar FBSDEs. An example is considered, using an approach based on stochastic flows that is related to the FBSDE approach, to further emphasize the parallels between the affine and quadratic models. An appendix discusses solvability and explicit solutions of the Riccati equations.

Keywords: Quadratic term-structure models; forward-backward stochastic differential equations; zero coupon bond price; quadratic price model; futures price; forward price; Riccati equations.

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1 Introduction

An important class of term-structure models are affine term-structure models (ATSMs). The defining characteristic of an ATSM is that the price at time $t \in [0, T]$ of a unit face value T -maturity zero-coupon bond, denoted by $P(t, T)$, is an exponential-affine function of an n -dimensional factor process X_t . That is, for times $0 \leq t \leq T$

$$P(t, T, X_t) = \exp(B(t, T)'X_t + C(t, T))$$

where $B(t, T)$ is an $n \times 1$ vector and $C(t, T)$ is a scalar. As a result, the yield of the bond is an affine function of the factor process. The class of ATSMs includes the models of Vašíček (1977), Cox, Ingersoll, and Ross (1985), Duffie and Kan (1996), Duffie et al. (2003), and many others. Despite several attractive properties ATSMs have been demonstrated to have some empirical limitations. For example, Dai and Singleton (2000) show that ATSMs fail to capture certain aspects of the distribution of swap yields, suggesting ATSMs may omit empirically observed nonlinearities. Further, Ahn and Gao (1999) demonstrate empirically that non-affine term structure models outperform one-factor affine models.

In order to address the limitations of ATSMs several authors have proposed the use of quadratic term-structure models (QTSMs). In a QTSM zero-coupon bond prices are exponential-quadratic functions of the factor process X_t for times $0 \leq t \leq T$

$$P(t, T, X_t) = \exp(X_t' A(t, T) X_t + B(t, T)' X_t + C(t, T))$$

where $A(t, T)$ is a non-singular $n \times n$ matrix, $B(t, T)$ is an $n \times 1$ vector, and $C(t, T)$ is a scalar. Ahn et al. (2002) introduce the comprehensive QTSMs and study the characteristics of these models. Pricing problems associated with QTSMs have been studied by Chen et al. (2004) and Leippold and Wu (2000). Other relevant research on QTSMs includes Levendorskiĭ (2005) and Boyarchenko and Levendorskiĭ (2007) which provide further evidence that QTSMs can capture nonlinearities between economic factors and provide more flexibility when constructing models when compared to ATSMs. Moreover, as shown by Chen et al. (2004) and Leippold and Wu (2000, 2002), QTSMs are analytically tractable as the prices of European style options can be calculated by Fourier transform methods similar to ATSMs. Gaspar (2004) also considers quadratic term structures for bond, futures, and forward prices.

In this paper we consider QTSMs using two nontraditional, but related, approaches to pricing problems. The first approach, and our main focus, is based on forward-backward stochastic differential equations (FBSDEs), which we henceforth refer to the *FBSDE approach* and was previously introduced in the context of ATSMs in Hyndman (2005, 2007a, 2009). By first characterizing the factor process and bond price in terms of the solution of coupled nonlinear FBSDEs and then demonstrating existence, uniqueness, and explicit solutions of the the FBSDEs the pricing problem is solved. The key result of the FBSDE approach is the extension of a technique due to Yong (1999) of a method for proving the existence, uniqueness, and explicit solution of certain coupled linear FBSDEs to the nonlinear FBSDEs which characterize the bond pricing problem in ATSMs. The same techniques were employed to characterize futures prices and forward prices in affine price models (APMs) in Hyndman (2009). In this paper we extend the FBSDE approach to the bond pricing problem in the context of QTSMs and futures and forward prices of quadratic price models (QPMs) for a risky asset. We obtain results characterizing these prices which are similar to the ATSM case and in particular provide new examples of quadratic FBSDEs with explicit solutions.

The second approach, which we briefly consider, is based on the stochastic flows method studied by Elliott and van der Hoek (2001), Grasselli and Tebaldi (2007), and Hyndman (2007b, 2009). This method gives a closed-form solution to the pricing problems for certain ATSMs. Geman and Yor (1993) and Yor (2010) have shown that the CIR process is a Bessel process under certain restrictions, which means that the CIR process and QTSMs are equivalent in certain cases. Motivated by this fact, we extend the techniques of the stochastic flows approach and the FBSDE approach to QTSMs.

The paper is organized as follows. Section 2 briefly introduces the modelling framework and notation. Section 3 reviews the FBSDE approach for the zero-coupon bond price and extends the results of Hyndman (2009) from ATSMs to QTSMs. Section 4 considers models where a risky asset price is an exponential quadratic function of the factor process (QPM), which includes the zero-coupon bond price, and applies the FBSDE approach to the futures price and forward price. Section 5 briefly considers the stochastic flows approach to QTSMs. We give an explicit solution for the zero-coupon bond price for our model in the one-dimensional case based on the flows method. Section 6 concludes and the Appendix considers the solvability of certain matrix Riccati-type differential equations which give sufficient conditions for the main results of this paper.

2 Preliminaries and notation

We shall begin our analysis on the risk-neutral filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$ for $0 \leq t \leq T^*$ where T^* is the fixed and finite investment horizon, $\{\mathcal{F}_t\}$ is a right-continuous and complete filtration satisfying the usual conditions, and Q is the risk-neutral (martingale) measure. Under these assumptions, as in Shreve (2004, p. 411), the price of the zero-coupon bond at time t for maturity $T \leq T^*$ is given by

$$P(t, T) = E_Q \left[\exp \left(- \int_t^T r_u du \right) \middle| \mathcal{F}_t \right] \quad (1)$$

where r_t is the instantaneous riskless interest rate. It is possible to calculate the conditional expectation (1) in several different ways after specifying the dynamics of the riskless interest rate.

On the risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$ assume that the riskless interest rate is a function of an \mathbb{R}^n -valued, $\{\mathcal{F}_t\}$ -adapted state process X_t given as the solution to the Gaussian stochastic differential equation (SDE)

$$dX_t = (AX_t + B)dt + \sigma dW_t, \quad X_0 = x_0 \quad (2)$$

with $A = [a_{i,j}]$ an $(n \times n)$ -matrix, $B = [\tilde{b}_i]'$ an $(n \times 1)$ -column vector, $\sigma = [\sigma_{i,j}]$ an $(n \times n)$ -matrix, and $W_t = [W_t^{(1)}, \dots, W_t^{(n)}]'$ a standard Brownian motion with respect to (\mathcal{F}_t, Q) . We assume that the riskless interest rate r_t is given by a quadratic function of the factors process.

Assumption 2.1 *The instantaneous riskless interest rate process is given by $r_t = r(X_t)$ where X_t is the strong solution to equation (2) and, for $x \in \mathbb{R}^n$,*

$$r(x) = x' \Gamma x + R x + k, \quad (3)$$

where $\Gamma = [\gamma_{i,j}]$ is a positive semidefinite $(n \times n)$ -matrix, $R = [r_1, \dots, r_n]$ is an $(1 \times n)$ -row vector, and k is a scalar such that

$$k \geq \frac{1}{4} R \Gamma^{-1} R'. \quad (4)$$

One of the common criticisms of Gaussian short rate models, such as the Vašíček (1977) model, is the potential for producing negative interest rates. However, in this case since Γ is positive semidefinite the lower bound of $r(x)$ from equation (3) is $(k - \frac{1}{4} R \Gamma^{-1} R')$ when $x = -\frac{1}{2} \Gamma^{-1} R'$. Therefore, the model given by Assumption 2.1, with the restriction of equation (4), produces a nonnegative instantaneous interest rate process $r_t = r(X_t)$ for $t \geq 0$.

Assumption 2.1 and equation (1) may now be used to extend the forward-backward stochastic differential equation (FBSDE) approach to term structure modelling for ATSMs of Hyndman (2009) to QTSMs. The main difference from Hyndman (2009) is that in this paper the dynamics of the factor process are given by a Gaussian (rather than affine) process and the riskless interest rate is a quadratic (rather than affine) functional of the factors process. The extension is motivated by the fact that the sum of squared components of an n -dimensional Ornstein-Uhlenbeck process can be identified with a CIR process as shown in Elliott and Kopp (2005, pp. 271-273) and more generally by the relationships between Brownian motion and squared Bessel (BESQ) processes described by Geman and Yor (1993) and Yor (2010).

3 Connections between QTSMs and FBSDEs

We briefly review the derivation of the forward-backward stochastic differential equation which characterizes the factor process and the bond price by considering the processes

$$H_s = \exp \left(- \int_0^s r(X_u) du \right)$$

and

$$V_s = E_Q \left[\exp \left(- \int_0^T r(X_u) du \right) \middle| \mathcal{F}_s \right] \quad (5)$$

for $s \in [0, T]$. Assumption 2.1 is in force throughout, however, provided the process V_s defined by equation (5) is a $(\mathcal{F}_t, \mathcal{Q})$ -martingale the characterization is valid more generally whatever the dynamics of the factors process or functional dependence of the risk-free rate on the factors.

Since H_s is \mathcal{F}_s -measurable from equation (1) it is clear that $P(s, T) = (V_s/H_s)$. By the Martingale Representation Theorem Shreve (2004, Theorem 5.4.2) there exists an \mathcal{F}_s -adapted process $J_s = [J_s^{(1)}, \dots, J_s^{(n)}]$, expressed as an $(1 \times n)$ vector process, such that

$$V_s = V_0 + \int_0^s J_u dW_u. \quad (6)$$

Since H_s is of finite variation and thus satisfies the dynamics

$$dH_s = -r(X_s)H_s ds.$$

Itô's formula gives that $Y_s = (V_s/H_s)$ satisfies

$$Y_s = Y_0 + \int_0^s r(X_u)Y_u du + \int_0^s Z_u dW_u$$

where $Z_u = (J_u/H_u)$. Subtracting the dynamics of Y_T from (3) we have

$$Y_s = Y_T - \int_s^T r(X_u)Y_u du - \int_s^T Z_u dW_u.$$

Since Y_T is the T -maturity zero-coupon bond price at time T , we have from equation (1) that $Y_T = P(T, T) = 1$. Therefore, on the risk neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathcal{Q})$, we have that for $s \in [0, T]$ the process (X_s, Y_s, Z_s) satisfies the system

$$X_s = X_0 + \int_0^s (AX_u + B) du + \int_0^s \sigma dW_u \quad (7)$$

$$Y_s = 1 - \int_s^T r(X_u)Y_u du - \int_s^T Z_u dW_u. \quad (8)$$

Equations (7)-(8) constitute a forward-backward stochastic differential equation (FBSDE). The characterization presented demonstrates the existence and uniqueness of an adapted solution (X_s, Y_s, Z_s) of the FBSDE (7)-(8) as defined in Pardoux and Peng (1992) (see also El Karoui et al. (1997) and Ma and Yong (1999) for a discussion of FBSDEs). However, due to the fact that Z arises from an application of the Martingale Representation Theorem the solution is not known in closed form. In order to derive a closed form solution we next apply a change of measure and consider the dynamics of the FBSDE under the new measure. Then, adapting a method of Yong (1999) for the case of linear FBSDEs to the resulting nonlinear FBSDE, we are able to prove existence and uniqueness as well as provide an explicit solution.

Recall the definition of the forward measure:

Definition 3.1 *Let the zero-coupon bond be the numéraire. Define*

$$\Lambda_T = P(0, T)^{-1} \exp \left(- \int_0^T r(X_u) du \right). \quad (9)$$

Then the T -forward measure Q^T is defined by

$$Q^T(A) := \int_A \Lambda_T dQ,$$

$\forall A \in \mathcal{F}_T$.

Define $\Lambda_t = E[\Lambda_T | \mathcal{F}_t]$ and note that

$$\Lambda_t = E \left[P(0, T)^{-1} \exp \left(- \int_0^T r(X_u) du \right) \middle| \mathcal{F}_t \right] = V_0^{-1} E \left[\exp \left(- \int_0^T r(X_u) du \right) \middle| \mathcal{F}_t \right] = V_0^{-1} V_t. \quad (10)$$

Substitute equation (10) into equation (6) to find

$$V_0 \Lambda_s = V_0 \Lambda_0 + \int_0^s J_u dW_u. \quad (11)$$

Dividing both sides of (11) by V_0 the dynamics of Λ_s are given by

$$\Lambda_s = \Lambda_0 + \int_0^s \frac{J_u}{V_0} dW_u = 1 + \int_0^s \frac{J_u}{V_0} \frac{H_u}{H_u} \frac{V_u}{V_u} dW_u = 1 + \int_0^s Y_u^{-1} Z_u \Lambda_u dW_u.$$

Then, by Girsanov's Theorem, the process $\{W_t^T\}_{0 \leq t \leq T}$ defined by

$$W_t^T = W_t - \int_0^t \frac{Z_u'}{Y_u} du$$

is a standard Brownian motion under the forward measure Q^T . Therefore, under the forward measure Q^T , the FBSDE (7)-(8) becomes

$$\begin{aligned} X_s &= X_0 + \int_0^s (AX_u + B + \sigma \frac{Z_u'}{Y_u}) du + \int_0^s \sigma dW_u^T \\ Y_s &= 1 - \int_s^T \left(Y_u r(X_u) + \frac{Z_u Z_u'}{Y_u} \right) du - \int_s^T Z_u dW_u^T \end{aligned}$$

for $s \in [0, T]$. In particular for the QTSM, with $r(X_t)$ given by (2.1), we have for $s \in [0, T]$

$$X_s = X_0 + \int_0^s (AX_u + B + \sigma \frac{Z_u'}{Y_u}) du + \int_0^s \sigma dW_u^T \quad (12)$$

$$Y_s = 1 - \int_s^T \left(Y_u [X_u' \Gamma X_u + R X_u + k] + \frac{Z_u Z_u'}{Y_u} \right) du - \int_s^T Z_u dW_u^T. \quad (13)$$

The FBSDE (12)-(13) is nonlinear, including two quadratic terms in the driver of the BSDE, and fully coupled. Further, given the quadratic term $(X_u' \Gamma X_u)$ in the driver of the BSDE it does not fall into the class considered in Hyndman (2009). According to Richter (2012) there are very few examples where explicit solutions to quadratic BSDEs are available. Obviously, the results of Hyndman (2005, 2007a, 2009) provide examples. We shall prove, independent of the construction already presented, that the FBSDE (12)-(13) provides another example. Similar to the results of Hyndman (2005, 2007a, 2009) the solution is explicit up to the solution of a Riccati type ODE. The Riccati equations and sufficient conditions for solvability are as follows.

Lemma 3.2 *If $\Gamma \in \mathbb{R}^{n \times n}$ is positive semidefinite the Riccati-type differential equations*

$$0 = \frac{d}{dt} R_2(t) + (R_2'(t) + R_2(t))A - \Gamma + \frac{1}{2} (R_2(t) + R_2'(t)) \sigma \sigma' (R_2(t) + R_2'(t)), \quad R_2(T) = 0_{n \times n} \quad (14)$$

$$0 = \frac{d}{dt} R_1(t) + R_1(t)A + B'(R_2(t) + R_2'(t)) - R + R_1(t) \sigma \sigma' (R_2(t) + R_2'(t)), \quad R_1(T) = 0_{1 \times n} \quad (15)$$

admit unique solutions $R_2(\cdot) \in \mathbb{R}^{n \times n}$, $R_1(\cdot) \in \mathbb{R}^{1 \times n}$ for all $t \in [0, T]$.

Proof: The nonsymmetric matrix Riccati equation (14) is a special case of the more general equations considered in the Appendix if we set $\Upsilon = \Gamma$ and $\Theta = 0_{n \times n}$ in equations (86)-(87) of Theorem A.1. Therefore, as Γ is positive semidefinite we have $(\Upsilon + \Upsilon')$ is positive semidefinite and $(\Theta + \Theta')$ is negative semidefinite. That is, the conditions of Theorem A.1 are satisfied so $R_2(t)$ given by (14) has a unique solution on the interval $[0, T]$. The solution of equation (15) then follows from Corollary A.4 with $\theta = 0$ and $\Psi = R$. ■

Considering the logarithm of the BSDE (13) simplifies the proof of the main result. By Itô's formula from s to T we have

$$\log Y_s = - \int_s^T \left\{ \frac{1}{2} \frac{Z_u Z_u'}{Y_u^2} + X_u' \Gamma X_u + R X_u + k \right\} du - \int_s^T \frac{Z_u}{Y_u} dW_u^T. \quad (16)$$

Theorem 3.3 *If $\Gamma \in \mathbb{R}^{n \times n}$ is positive semidefinite the FBSDE (12)-(13) admits a unique adapted solution (X_t, Y_t, Z_t) for all $t \in [0, T]$ given by*

$$X_t = X_0 + \int_0^t \{ [A + \sigma \sigma' (R_2(u) + R_2'(u))] X_u + B + \sigma \sigma' R_1'(u) \} du + \int_0^t \sigma dW_u^T, \quad (17)$$

$$Y_t = \exp \left(X_t' R_2(t) X_t + R_1(t) X_t + R_0(t) \right), \quad (18)$$

$$Z_t = [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma Y_t \quad (19)$$

where $R_2(t)$ is the solution of equation (14), $R_1(t)$ is the solution of equation (15), and $R_0(t)$ is given by

$$R_0(t) = - \int_t^T \left(k - R_1(s) B - \frac{1}{2} R_1(s) \sigma \sigma' [R_1(s)]' - \text{Tr}(\sigma' R_2(s) \sigma) \right) ds. \quad (20)$$

Proof: First, we must show that (X, Y, Z) given by (17)-(19) satisfy the FBSDE (12)-(13). Dividing (19) by (18), we have

$$\frac{Z_t}{Y_t} = [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma. \quad (21)$$

Substituting (21) into (17), we obtain the dynamics of X_t given by equation (12). So X_t given by (17) satisfies the SDE (12).

Consider the function $f(t, x) = \exp \left(x' R_2(t) x + R_1(t) x + R_0(t) \right)$. Applying Itô's formula to $f(t, x)$ using the dynamics of X_t in (17) we find that $Y_t = f(t, X_t)$ and Z_t defined by (18) and (19) satisfy

$$\begin{aligned} dY_t &= \{ X_t' \dot{R}_2(t) X_t + \dot{R}_1(t) X_t + \dot{R}_0(t) + X_t' (R_2(t) + R_2'(t)) A X_t \\ &\quad + B' (R_2(t) + R_2'(t)) X_t + [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma \frac{Z_t'}{Y_t} + R_1(t) A X_t + R_1(t) B + \text{Tr}(\sigma' R_2(t) \sigma) \\ &\quad + \frac{1}{2} [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma \sigma' [(R_2'(t) + R_2(t)) X_t + R_1'(t)] \} Y_t dt \\ &\quad + Y_t [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma dW_t^T \end{aligned} \quad (22)$$

where $\dot{R}_k(t) = \frac{d}{dt} R_k(t)$, $k = 0, 1, 2$. Substitute (21) in (22) to find

$$\begin{aligned} dY_t &= \{ X_t' [\dot{R}_2(t) + (R_2(t) + R_2'(t)) A + \frac{1}{2} (R_2(t) + R_2'(t)) \sigma \sigma' (R_2'(t) + R_2(t))] X_t \\ &\quad + [\dot{R}_1(t) + B' (R_2(t) + R_2'(t)) + R_1(t) A + R_1(t) \sigma \sigma' (R_2'(t) + R_2(t))] X_t \\ &\quad + \frac{Z_t Z_t'}{Y_t^2} + \dot{R}_0(t) + R_1(t) B + \text{Tr}(\sigma' R_2(t) \sigma) + \frac{1}{2} R_1(t) \sigma \sigma' R_1'(t) \} Y_t dt + Z_t dW_t^T. \end{aligned} \quad (23)$$

Substituting equations (14)-(15) and (20) into equation (23) gives that Y_t defined by (17)-(19) satisfies

$$Y_t = Y_T - \int_t^T \{ Y_u [X_u' \Gamma X_u + R X_u + k] + \frac{Z_u Z_u'}{Y_u} \} du - \int_t^T Z_u dW_u^T.$$

By the boundary conditions of (14)-(15) and (20) at $t = T$ we have, from equation (18), that

$$Y_T = \exp (X_T' R_2(T) X_T + R_1(T) X_T + R_0(T)) = \exp (X_T' 0_{n \times n} X_T + 0_{1 \times n} X_T + 0) = 1.$$

Hence (X, Y, Z) given by (17)-(19) satisfy the FBSDE (12)-(13).

Second, we prove the uniqueness of the solution. Let (X, Y, Z) be any adapted solution of the FBSDE (12)-(13). Define

$$\begin{aligned}\log \bar{Y}_t &= X_t' R_2(t) X_t + R_1(t) X_t + R_0(t), \\ \bar{Z}_t &= [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma \bar{Y}_t,\end{aligned}$$

then

$$\frac{\bar{Z}_t}{\bar{Y}_t} = [X_t' (R_2(t) + R_2'(t)) + R_1(t)] \sigma. \quad (24)$$

Apply Itô's formula to $f(t, x) = x' R_2(t) x + R_1(t) x + R_0(t)$, where X_t is given by (12) to find

$$\begin{aligned}df(t, X_t) &= d \log \bar{Y}_t = \{X_t' \dot{R}_2(t) X_t + \dot{R}_1(t) X_t + \dot{R}_0(t) \\ &\quad + X_t' (R_2'(t) + R_2(t)) A X_t + R_1(t) A X_t + B' (R_2(t) + R_2'(t)) X_t + R_1(t) B \\ &\quad + [X_t' (R_2'(t) + R_2(t)) + R_1(t)] \sigma \frac{Z_t'}{Y_t} + Tr(\sigma' R_2(s) \sigma)\} dt + [X_t' (R_2'(t) + R_2(t)) + R_1(t)] \sigma dW_t^T.\end{aligned} \quad (25)$$

Substitute equations (14), (15), (20) and (24) into equation (25) to find

$$d \log \bar{Y}_t = \{X_t' \Gamma X_t + R X_t + k - \frac{1}{2} \frac{\bar{Z}_t \bar{Z}_t'}{\bar{Y}_t^2} + \frac{\bar{Z}_t Z_t'}{\bar{Y}_t Y_t}\} dt + \frac{\bar{Z}_t}{\bar{Y}_t} dW_t^T$$

and $\log \bar{Y}_T = 0$. So

$$\log \bar{Y}_t = - \int_t^T \{X_u' \Gamma X_u + R X_u + k - \frac{1}{2} \frac{\bar{Z}_u \bar{Z}_u'}{\bar{Y}_u^2} + \frac{\bar{Z}_u Z_u'}{\bar{Y}_u Y_u}\} du - \int_t^T \frac{\bar{Z}_u}{\bar{Y}_u} dW_u^T. \quad (26)$$

Subtract equation (26) from equation (16) to find

$$\begin{aligned}\log Y_t - \log \bar{Y}_t &= - \int_t^T \left\{ \frac{1}{2} \frac{Z_u Z_u'}{Y_u^2} - \frac{\bar{Z}_u Z_u'}{\bar{Y}_u Y_u} + \frac{1}{2} \frac{\bar{Z}_u \bar{Z}_u'}{\bar{Y}_u^2} \right\} du - \int_t^T \left\{ \frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u} \right\} dW_u^T \\ &= - \frac{1}{2} \int_t^T \left\{ \left(\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u} \right) \left(\frac{Z_u'}{Y_u} - \frac{\bar{Z}_u'}{\bar{Y}_u} \right) \right\} du - \int_t^T \left(\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u} \right) dW_u^T.\end{aligned} \quad (27)$$

Define

$$\begin{aligned}\hat{Y}_t &= \log Y_t - \log \bar{Y}_t \\ \hat{Z}_t &= \frac{Z_t}{Y_t} - \frac{\bar{Z}_t}{\bar{Y}_t}.\end{aligned}$$

Then equation (27) becomes

$$\hat{Y}_t = - \frac{1}{2} \int_t^T \hat{Z}_u \hat{Z}_u' du - \int_t^T \hat{Z}_u dW_u^T. \quad (28)$$

By the result of Kobylanski (2000, Theorem 2.3), the BSDE (28) admits the unique adapted solution $(\hat{Y}_t, \hat{Z}_t) = (0, 0_{1 \times n})$. So we have $Y_t = \bar{Y}_t$ and $Z_t = \bar{Z}_t$. This means that any adapted solution (X, Y, Z) of the FBSDE (12)-(13) must satisfy (17), (18) and (19). \blacksquare

Since $Y_t = P(t, T)$ a simple corollary of Theorem 3.3 gives that the zero coupon bond price is an exponential quadratic function of the factor process.

Corollary 3.4 *If the factor process is given by (2) and the short rate process is represented by Assumption 2.1, then the zero coupon bond price has an exponential quadratic form,*

$$P(t, T) = \exp\{X_t' R_2(t) X_t + R_1(t) X_t + R_0(t)\},$$

where $R_2(t)$, $R_1(t)$ and $R_0(t)$ solve equations (14), (15) and (20), respectively.

Remark 3.5 *The existence and uniqueness of the solution to Riccati-type differential equations (14)-(15) and (20) is similar to that of Gombani and Runggaldier (2013) where the QTSM is characterized in terms of a linear-quadratic control (LQC) problem. Indeed, much more is known about the solvability of the Riccati equations associated with LQC problems than those arising in ATSMs. We defer a detailed discussion of the explicit solvability of the Riccati equations arising in the QTSM problem to the Appendix where we consider more general equations which include (14)-(15), as well as those arising in our results on futures and forward prices, as special cases.*

We next consider a model of risky asset prices that allows us to apply similar techniques to characterize the futures price and forward price of the asset.

4 Quadratic Price Models

Consider a risky asset with price given as an exponential quadratic function of the factors process (2). This class of price processes allows for the consideration of futures and forward contracts on zero-coupon bonds where the bond price is given as in Corollary 3.4.

Assumption 4.1 *Assume that the risk-neutral dynamics of the factor process are given by equation (2) and that the price of the risky asset S at time $t \in [0, T]$ is given by $S(t, X_t)$ where*

$$S(t, x) = \exp(x' a(t)x + b(t)x + c(t)) \quad (29)$$

where $a(t) \in \mathbb{R}^{n \times n}$, $b(t) \in \mathbb{R}^{1 \times n}$, and $c(t) \in \mathbb{R}$ are continuous on the interval $[0, T]$ and $a(T)$ is negative semidefinite.

Equation (29) defines a quadratic price model (QPM). We next extend the results of the previous section and Hyndman (2009) to consider the futures price and forward price associated with a QPM.

4.1 Futures Prices

Consider the T -futures price of the risky asset S at time $t \in [0, T]$ defined by

$$G(t, T) = E_Q[S(T, X_T) | \mathcal{F}_t]$$

where $S(t, x)$ is given by equation (29). Similar to the results of the previous section and the results of Hyndman (2009) we may characterize the factor process and futures price as the solution to a FBSDE.

Define $Y_t = G(t, T)$ so that, by the Martingale Representation Theorem, there exists an \mathcal{F}_t -adapted process $Z_t = [Z_t^1, \dots, Z_t^n]$ such that

$$Y_t = Y_0 + \int_0^t Z_u dW_u. \quad (30)$$

Note that Z_u is an $(1 \times n)$ -vector valued process. Therefore,

$$Y_t - Y_T = - \int_t^T Z_u dW_u.$$

Since $Y_T = G(T, T) = S(T, X_T)$ we have the following BSDE for the futures price

$$Y_t = S(T, X_T) - \int_t^T Z_u dW_u.$$

Take $N(\cdot) = \exp(\int_0^\cdot r(X_u) du) G(\cdot, T)$ as numéraire and define the measure P^G by the Radon-Nikodym derivative

$$\Lambda_T = \frac{dP^G}{dQ} \Big|_{\mathcal{F}_T} = e^{-\int_0^T r(X_u) du} \frac{N(T)}{N(0)} = \frac{G(T, T)}{G(0, T)} = \frac{S(T, X_T)}{G(0, T)}. \quad (31)$$

Define $\Lambda_t = E_Q[\Lambda_T | \mathcal{F}_t]$ and note from equations (30) and (31) that

$$\Lambda_t = E_Q \left[\frac{S(T, X_T)}{G(0, T)} \middle| \mathcal{F}_t \right] = \frac{G(t, T)}{G(0, T)} = \frac{Y_t}{Y_0}.$$

Dividing equation (30) by Y_0 gives that the dynamics of Λ_t are

$$\Lambda_t = 1 + \int_0^t \frac{Z_u}{Y_0} dW_u = 1 + \int_0^t \frac{Y_u}{Y_0} \frac{Z_u}{Y_u} dW_u = 1 + \int_0^t \Lambda_u \frac{Z_u}{Y_u} dW_u.$$

Girsanov's Theorem then gives that the process $\{W_t^G\}_{0 \leq t \leq T}$ defined by

$$W_t^G = W_t - \int_0^t \frac{Z_u}{Y_u} du$$

is a standard (\mathcal{F}_t, P^G) -Brownian motion. Writing the dynamics of X_t and Y_t under P^G we obtain, similar to Hyndman (2009) the following quadratic FBSDE for the futures price

$$X_t = X_0 + \int_0^t \left[AX_u + B + \sigma \frac{Z_u}{Y_u} \right] du + \int_0^t \sigma dW_u^G \quad (32)$$

$$Y_t = S(T, X_T) - \int_t^T \frac{Z_u Z_u'}{Y_u} du - \int_t^T Z_u dW_u^G. \quad (33)$$

While the dynamics of X are Gaussian, and hence a special case of those considered in Hyndman (2007b, 2009), the dynamics for Y differ due to the exponential quadratic form of the terminal condition. Nevertheless, following the methodology of Hyndman (2009) and the previous section we are able to prove the following result which gives an explicit solution to the coupled quadratic FBSDE (32)-(33). The proof, which is independent of the construction of the FBSDE, is similar to the proof of Theorem 3.3 and the results of Hyndman (2009) and is therefore omitted. We first present sufficient conditions for the solvability of the Riccati equations.

Lemma 4.2 *If $a(T) \in \mathbb{R}^{n \times n}$ is negative semidefinite the Riccati-type differential equations*

$$0 = \frac{d}{dt} R_2^G(t) + \left([R_2^G(t)]' + R_2^G(t) \right) A + \frac{1}{2} \left([R_2^G(t)]' + R_2^G(t) \right) \sigma \sigma' \left([R_2^G(t)]' + R_2^G(t) \right), \quad R_2^G(T) = a(T) \quad (34)$$

$$0 = \frac{d}{dt} R_1^G(t) + R_1^G(t) \left\{ A + \sigma \sigma' \left([R_2^G(t)]' + R_2^G(t) \right) \right\} + B' \left([R_2^G(t)]' + R_2^G(t) \right), \quad R_1^G(T) = b(T) \quad (35)$$

admit unique solutions $R_2^G(t) \in \mathbb{R}^{n \times n}$ and $R_1^G(t) \in \mathbb{R}^{1 \times n}$ for all $t \in [0, T]$.

Proof: The result follows, similar to the proof of Lemma 3.2, if we set $\Upsilon = 0$ and $\Theta = a(T)$ in equations (86)-(87) of Theorem A.1 and $\Psi = 0$ and $\theta = b(T)$ in Corollary A.4. ■

Theorem 4.3 *If $a(T) \in \mathbb{R}^{n \times n}$ is negative semidefinite and $S(t, x)$ is given by equation (29) the FBSDE (32)-(33) has a unique adapted solution (X, Y, Z) given by*

$$\begin{aligned} X_t &= X_0 + \int_0^t \left\{ \left(A + \sigma \sigma' \left([R_2^G(u)]' + R_2^G(u) \right) \right) X_u + \left(B + \sigma \sigma' [R_1^G(u)]' \right) \right\} du + \int_0^t \sigma dW_u^G \\ Y_t &= \exp \left(X_t' R_2^G(t) X_t + R_1^G(t) X_t + R_0^G(t) \right) \\ Z_t &= \left[X_t' \left([R_2^G(t)]' + R_2^G(t) \right) + R_1^G(t) \right] \sigma Y_t \end{aligned}$$

where $R_2^G(t)$ is the solution of equation (34), $R_1^G(t)$ is the solution of equation (35), and

$$R_0^G(t) = c(T) + \int_t^T \left(R_1^G(u) B + \frac{1}{2} R_1^G(u) \sigma \sigma' [R_1^G(u)]' + \text{Tr} \left(\sigma' R_2^G(u) \sigma \right) \right) du. \quad (36)$$

Corollary 4.4 *Under the conditions of Assumption 4.1 the futures price has the exponential quadratic form*

$$G(t, T) = \exp \left(X_t' R_2^G(t) X_t + R_1^G(t) X_t + R_0^G(t) \right)$$

where $R_2^G(t)$ and $R_1^G(t)$ are solutions to equations (34)-(35) and $R_0^G(t)$ is given by (36).

Remark 4.5 *Note that the Riccati equations (34)-(35) are similar to those associated with the bond price. We defer discussion of the existence, uniqueness, and explicit solution of more general equations which include (34)-(35) as special cases to the Appendix.*

We next consider the forward price of the risky asset.

4.2 Forward Prices

Recall that the T -forward price of the risky asset at time $t \in [0, T]$ is

$$F(t, T) = \frac{E_Q[\exp \left(- \int_t^T r(X_u) du \right) S(T, X_T) | \mathcal{F}_t]}{P(t, T)} \quad (37)$$

where $P(t, T)$ is the price at time t of the zero coupon bond maturing at time T . In the case that the risk free interest rate is deterministic the futures and forward prices are identical. Therefore, we shall assume that the interest rate is given as in Section 3 and the risky asset price S is given by equation (29). Further, we assume that the risky asset pays a dividend yield (or convenience yield) so that the discounted asset price is not a Q -martingale which would reduce the numerator of equation (37) to the risky asset price.

Similar to Section 3 in the case of the bond price and Hyndman (2009) in the case of the forward price in an APM we characterize the factor process and the numerator of equation (37), which is the risk neutral present value of a forward commitment to deliver one unit of the risky asset at time T , in terms of a FBSDE. Define

$$V_s = E_Q[\exp \left(- \int_0^T r(X_u) du \right) S(T, X_T) | \mathcal{F}_s]$$

and

$$H_s = \exp \left(- \int_0^s r(X_u) du \right).$$

Let $Y_s = V_s/H_s$ and note that $Y_s = F(s, T)P(s, T)$.

Since V_s is a martingale we have, by the Martingale Representation Theorem, that there exists an adapted process $J_t = [J_t^1, \dots, J_t^n]$, represented as a $(1 \times n)$ vector valued process, such that

$$V_s = V_0 + \int_0^s J_u dW_u. \quad (38)$$

Apply Itô's formula to find that Y_t satisfies the BSDE

$$Y_t = S(T, X_T) - \int_t^T r(X_u) Y_u du - \int_t^T Z_u dW_u \quad (39)$$

where $Z_u = J_u/H_u$.

Define the risk-neutral measure for the numéraire $N(\cdot) = F(\cdot, T)P(\cdot, T)$, denoted by Q^F , by the Radon-Nikodym derivative

$$\Gamma_T = \frac{dQ^F}{dQ} \Big|_{\mathcal{F}_T} = \frac{F(T, T)P(T, T)}{F(0, T)P(0, T)} \exp \left(- \int_0^T r(X_u) du \right) = \frac{S(T, X_T)}{F(0, T)P(0, T)} \exp \left(- \int_0^T r(X_u) du \right).$$

Then, with $\Gamma_t = E_Q[\Gamma_T | \mathcal{F}_t]$, we have that $\Gamma_t = V_t/V_0$ and, by equation (38),

$$\Gamma_t = 1 + \int_0^t \Gamma_u \frac{Z_u}{Y_u} dW_u.$$

Hence, by Girsanov's theorem,

$$W_t^F = W_t - \int_0^t \frac{Z_u'}{Y_u} du \quad (40)$$

is an (\mathcal{F}_t, Q^F) -Brownian motion. Using equation (40) to write the dynamics of (X, Y) , given by equations (7) and (39), under the measure Q^F we obtain the following coupled quadratic FBSDE

$$X_t = X_0 + \int_0^t \left(AX_u + B + \sigma \frac{Z_u'}{Y_u} \right) du + \int_0^t \sigma dW_u^F \quad (41)$$

$$Y_t = S(T, X_T) - \int_t^T \left(r(X_u)Y_u + \frac{Z_u Z_u'}{Y_u} \right) du - \int_t^T Z_u dW_u^F. \quad (42)$$

Note that the FBSDE (41)-(42) is similar to the FBSDE presented in Hyndman (2009) for APMs except that the volatility dynamics of X are simpler while the functions r and S are, as functions of X_t , quadratic and exponential-quadratic functions rather than affine and exponential-affine functions respectively. Similar to the result of Section 3 for the case of the bond and Hyndman (2009) for APMs the following result gives an explicit solution to the quadratic FBSDE (41)-(42), which is independent of the construction presented, and hence we omit the proof.

Lemma 4.6 *If $\Gamma \in \mathbb{R}^{n \times n}$ is positive semidefinite and $a(T) \in \mathbb{R}^{n \times n}$ is negative semidefinite then the Riccati-type differential equations*

$$0 = \frac{d}{dt} R_2^F(t) + \left([R_2^F(t)]' + R_2^F(t) \right) A + \frac{1}{2} \left([R_2^F(t)]' + R_2^F(t) \right) \sigma \sigma' \left([R_2^F(t)]' + R_2^F(t) \right) - \Gamma, \quad R_2^F(T) = a(T) \quad (43)$$

$$0 = \frac{d}{dt} R_1^F(t) + R_1^F(t) A + B' \left([R_2^F(t)]' + R_2^F(t) \right) + R_1^F(t) \sigma \sigma' \left([R_2^F(t)]' + R_2^F(t) \right) - R, \quad R_1^F(T) = b(T) \quad (44)$$

admit unique solutions $R_2^F(t) \in \mathbb{R}^{n \times n}$ and $R_1^F(t) \in \mathbb{R}^{1 \times n}$ for all $t \in [0, T]$.

Proof: The result follows, similar to the proof of Lemma 3.2, if we set $\Upsilon = \Gamma$ and $\Theta = a(T)$ in equations (86)-(87) of Theorem A.1 and $\Psi = R$ and $\theta = b(T)$ in Corollary A.4. ■

Theorem 4.7 *If $\Gamma \in \mathbb{R}^{n \times n}$ is positive semidefinite, $a(T) \in \mathbb{R}^{n \times n}$ is negative semidefinite, $r(x)$ is given by equation (2), and $S(t, x)$ is given by equation (29) then the FBSDE (41)-(42) has a unique adapted solution (X, Y, Z) given by*

$$X_t = X_0 + \int_0^t \left\{ \left[A + \sigma \sigma' \left([R_2^F(u)]' + R_2^F(u) \right) \right] X_u + \left(B + \sigma \sigma' [R_1^F(u)]' \right) \right\} du + \int_0^t \sigma dW_u^F \quad (45)$$

$$Y_t = \exp \left(X_t R_2^F(t) X_t + R_1^F(t) X_t + R_0^F(t) \right) \quad (46)$$

$$Z_t = \left[X_t' \left([R_2^F(t)]' + R_2^F(t) \right) X_t + [R_1^F(t)]' \right] \sigma Y_t \quad (47)$$

where $R_2^F(t)$ is the solution of equation (43), $R_1^F(t)$ is the solution of equation (44), and

$$R_0^F(t) = c(T) - \int_t^T \left(k - R_1^F(u) B - \frac{1}{2} R_1^F(u) \sigma \sigma' [R_1^F(u)]' + \text{Tr} \left(\sigma' [R_2^F(u)]' \sigma \right) \right) du. \quad (48)$$

Corollary 4.8 *Under the conditions of Assumptions 2.1 and 4.1 the forward price is an exponential quadratic function of the factors process*

$$F(t, T) = \frac{\exp \left(X_t' R_2^F(t) X_t + R_1^F(t) X_t + R_0^F(t) \right)}{P(t, T)} = \frac{\exp \left(X_t' R_2^F(t) X_t + R_1^F(t) X_t + R_0^F(t) \right)}{\exp \left(X_t' R_2(t) X_t + R_1(t) X_t + R_0(t) \right)}$$

where $R_2^F(t)$ and $R_1^F(t)$ are the solutions to equations (43)-(44); $R_0^F(t)$ is given by equation (48); and $R_2(t)$, $R_1(t)$, and $R_0(t)$ are the solutions to equations (14), (15) and (20).

Remark 4.9 Comparing Theorems 3.3, 4.3, and 4.7 we see that the Riccati type equations (43)-(44) for $R_2^F(t)$ and $R_1^F(t)$ and the integral for $R_0^F(t)$ include as special cases the corresponding terms for the bond and futures prices if we make certain parametric restrictions. This general form is similar to the Riccati type differential equations of linear quadratic control (LQC). Further, these results suggest a correspondence between our results and the results of Gombani and Runggaldier (2013) which are based on LQC. We discuss the explicit solvability of a general Riccati type equation which includes (43)-(44) as a special case, and the correspondence to those of LQC in the Appendix.

In the next section we briefly consider an example of the application to QTSMs of the method of stochastic flows due to Elliott and van der Hoek (2001) which was originally developed in the context of ATSMs. A general formulation of the method of stochastic flows for QTSMs similar to that presented for ATSMs in Section 4 of Hyndman (2009) can be developed for QTSMs and extended to QPMs similar to the extension for presented in Sections 5.2 and 5.4 of Hyndman (2009) for APMs. However, the main motivation for our consideration of the flows method in the context of quadratic models driven by Gaussian factors, in contrast to the case for affine models driven by Gaussian factors which was considered in Elliott and van der Hoek (2001) and Hyndman (2007b), is to show that a measure change is necessary for the method to be effective. This objective can easily be accomplished by consideration of an example in the one-dimensional case.

5 Stochastic Flows

The FBSDE approach to characterizing the bond price in ATSM introduced in Hyndman (2009), and extended to QTSMs in this paper, was motivated by the stochastic flow approach introduced by Elliott and van der Hoek (2001). The stochastic flow approach expresses the price at time t of the T -maturity zero coupon bond as a function of the value x at time t of the factor process, and this dependence is denoted $P(t, T, x)$. By taking the derivative of $P(t, T, x)$ with respect to the initial condition, and then using properties of stochastic flows and their Jacobians, it is possible to express $P(t, T, x)$ as an ordinary differential equation (ODE) which illuminates the nature of the functional dependence of the bond price on the factor process. In Elliott and van der Hoek (2001) when the factors process is Gaussian and the interest rate is an affine function of the factors process the fact that the derivative of the factor process with respect to x is deterministic leads to a linear ODE for $P(t, T, x)$. In this case it is then immediate that the bond price is an exponential affine function of the factor process. These results were extended to characterize futures and forward prices on an asset which is an exponential affine function of a Gaussian factor process in Hyndman (2007b).

However, in Elliott and van der Hoek (2001) when the dynamics of the factors are given by an affine process and the interest rate is an affine function of the factors the fact that the derivative with respect to x of the factor process is not deterministic requires a change of measure in order to obtain a linear ordinary differential equation satisfied by the bond price. The coefficient of this linear ODE is the conditional expectation of a function of the derivative of the factor process. In the one-dimensional case, corresponding to the CIR model, semi-group properties of the stochastic flow can be used to show that this coefficient is deterministic. Unfortunately in the case that the factor process is multi-dimensional there was a gap in the proof of a key approximation lemma of Elliott and van der Hoek (2001) which was used to claim that the coefficient of the linear ODE is deterministic (see Hyndman (2009, 2011)). The FBSDE approach of Hyndman (2009) proved, as a corollary to the main results, that the coefficient of the linear ODE for the bond price in the stochastic flows approach is in fact deterministic.

In the case of a Gaussian factor process where the interest rate is a quadratic function of the factor process it is illustrative to consider the stochastic flow approach in the one-dimensional case. We find that although the factor process is Gaussian, and the derivative of the flow is deterministic, that the change of measure is still necessary in order to obtain an ODE for the bond price.

Suppose the dynamics of the factor process, X_t , are given on the risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathcal{Q})$ by the one-dimensional model

$$dX_t = \beta(\alpha - X_t)dt + \sigma dW_t \quad (49)$$

and the riskless interest rate r_t is given by the quadratic function of the factor process

$$r(X_t) = cX_t^2 + bX_t + a. \quad (50)$$

Write $X_s^{t,x}$ for the solution of (49) started from $x \in \mathbf{R}$ at time $t \geq 0$. That is, $X_s^{t,x}$ satisfies

$$X_s^{t,x} = x + \int_t^s \beta(\alpha - X_u^{t,x}) du + \sigma \int_t^s dW_u, \quad s \in [t, T].$$

We refer to $X_s^{t,x}$ as the *stochastic flow* associated with the factors process. Stochastic flows have been studied extensively by many authors including Elworthy (1978), Bismut (1981a,b), and Kunita (1984, 1990). The map $x \rightarrow X_s^{t,x}$ is \mathcal{Q} -almost surely differentiable and the derivative satisfies

$$\frac{\partial X_s^{t,x}}{\partial x} = 1 - \beta \int_t^s \frac{\partial X_u^{t,x}}{\partial x} du, \quad s \geq t \quad (51)$$

Protter (1990, Theorem 39, p 250). Equation (51) can be solved independent of x . If D_{ts} satisfies

$$D_{ts} = 1 - \beta \int_t^s D_{tu} du \quad (52)$$

then the unique solution to (52) is the exponential

$$D_{ts} = e^{-\beta(s-t)}. \quad (53)$$

That is, $D_{ts} = \frac{\partial X_s^{t,x}}{\partial x}$ which does not depend on x for all $0 \leq t \leq s \leq T$.

By the Markov property of X_t , $P(t, T) = P(t, T, X_t)$, where

$$P(t, T, x) = E \left[\exp \left\{ - \int_t^T r(X_u^{t,x}) du \right\} \right]. \quad (54)$$

Taking the derivative of (54) with respect to x , following Elliott and van der Hoek (2001) and Hyndman (2007a,b, 2009), we find

$$\begin{aligned} \frac{\partial P(t, T, x)}{\partial x} &= E \left[\left(- \int_t^T r'(X_u^{t,x}) \frac{\partial X_u^{t,x}}{\partial x} du \right) \exp \left(- \int_t^T r(X_u^{t,x}) du \right) \right] \\ &= E \left[L(t, T, x) \exp \left(- \int_t^T r(X_u^{t,x}) du \right) \right], \end{aligned} \quad (55)$$

where

$$L(t, T, x) = - \int_t^T (2cX_u^{t,x} + b) D_{tu} du.$$

We may exchange the order of expectation and differentiation since $b(x, t) := \beta(\alpha - x)$ and $\sigma(x, t) := \sigma$ satisfy linear growth conditions in x and a global Lipschitz condition, the partial derivatives of $b(x, t)$ and $\sigma(x, t)$ are continuous and satisfy a polynomial growth condition, and the function $\exp \left(- \int_t^T r(x) du \right)$ has two continuous derivatives satisfying a polynomial growth condition (see Friedman (1975, pp. 117-123)).

In contrast to the work of Elliott and van der Hoek (2001) and Hyndman (2007b), in the case of ATSMs driven by Gaussian factors, we may not factor $L(t, T, x)$ from the expectation in equation (55) as this function depends on $X_u^{t,x}$. Therefore, even though the model of the instantaneous interest rate given by Assumption 2.1, is driven by Gaussian factors we introduce a change of probability measure as in Elliott and van der Hoek (2001) and Hyndman (2007a, 2009). That is, in order formulate equation (55) as an ODE, we express the conditional expectation under the forward measure. This choice is motivated by the relationship between a squared Gaussian process and the CIR process previously mentioned.

Recall Definition 3.1 for the forward measure and note that for any \mathcal{F}_T -measurable random variable ϕ with $E_T|\phi| < \infty$ a general version of Bayes' Theorem (see, Karatzas and Shreve (1991, Lemma 3.5.3)) gives that

$$E_T[\phi|\mathcal{F}_t] = \Lambda_t^{-1} E[\phi\Lambda_T|\mathcal{F}_t] \quad (56)$$

where Λ_T is given by equation (9) and from equation (10) we may write

$$\Lambda_t = \frac{\exp\left(-\int_0^t r(X_u)du\right)P(t,T)}{P(0,T)}. \quad (57)$$

Substitute equations (9) and (57) into equation (56), noting that $\exp\left(\int_0^t r(X_u)du\right)$ is \mathcal{F}_t -measurable, to find

$$E_T[\phi|\mathcal{F}_t]P(t,T) = E\left[\phi \cdot \exp\left\{-\int_t^T r(X_u^{t,X_t})du\right\} \middle| \mathcal{F}_t\right]. \quad (58)$$

Therefore, with $\phi = L(t,T,X_t)$ in equation (58) we find

$$\left.\frac{\partial P(t,T,x)}{\partial x}\right|_{x=X_t} = P(t,T,X_t)E_T\left[L(t,T,X_t) \middle| \mathcal{F}_t\right] \quad (59)$$

which resembles a linear ODE for the bond price.

In order to solve equation (59) we must first consider the conditional expectation's dependence on X_t . To derive the dynamics of $X_u^{t,x}$ under the forward measure we must characterize the Brownian motion under Q^T using Girsanov's Theorem. As in Elliott and van der Hoek (2001) we may show that

$$\Lambda_t = \Lambda_0 - \int_0^t \Theta_u \Lambda_u dW_u,$$

where

$$\Theta_u = -\sigma E_T\left[L(u,T,X_u) \middle| \mathcal{F}_u\right]. \quad (60)$$

Therefore, by Girsanov's Theorem, the process W_t^T defined by

$$W_t^T = W_t + \int_0^t \Theta_u du \quad (61)$$

is a standard Brownian Motion with respect to the forward measure Q^T .

Using equations (60)-(61) the dynamics of $X_s^{t,x}$ under the forward measure are

$$X_s^{t,x} = x + \int_t^s \{\beta(\alpha - X_v^{t,x}) - \sigma\Theta_v\}dv + \sigma \int_t^s dW_v^T. \quad (62)$$

Apply Itô's product rule to the dynamics of D_{ts} given by (52) and $X_s^{t,x}$ given by (62). Then, since $D_{ts}(x)$ is of finite variation, we have

$$\begin{aligned} X_s^{t,x} D_{ts} &= x + \int_t^s D_{tv} dX_v^{t,x} + \int_t^s X_v^{t,x} dD_{tv} \\ &= x + \alpha\beta \int_t^s D_{tv} dv - 2\beta \int_t^s X_v^{t,x} D_{tv} dv + \sigma \int_t^s D_{tv} dW_v^T - \sigma^2 \int_t^s D_{tv} E_T\left[\int_v^T (2cX_{v_1}^{v,x} + b)D_{vv_1} \middle|_{x=X_v} dv_1 \middle| \mathcal{F}_v\right] dv. \end{aligned} \quad (63)$$

Evaluate equation (63) at $x = X_t$ and take the \mathcal{F}_t -conditional expectation under the forward measure Q^T to find

$$E_T[X_s^{t,X_t} D_{ts} | \mathcal{F}_t] = X_t + \int_t^s \left\{ \alpha \beta D_{tv} - 2\beta E_T[X_v^{t,X_t} D_{tv} | \mathcal{F}_t] - \sigma^2 \int_v^T E_T[D_{tv} E_T[(2cX_{v_1}^{v,X_v} + b) D_{vv_1} dv_1 | \mathcal{F}_v] | \mathcal{F}_t] \right\} dv. \quad (64)$$

By the tower property of conditional expectation, since $t \leq v \leq s \leq T$, the conditional expectation in the double integral of equation (64) becomes

$$E_T[D_{tv} E_T[(2cX_{v_1}^{v,X_v} + b) D_{vv_1} | \mathcal{F}_v] | \mathcal{F}_t] = E_T[D_{tv}(2cX_{v_1}^{v,X_v} + b) D_{vv_1} | \mathcal{F}_t] = 2cE_T[D_{tv} D_{vv_1} X_{v_1}^{v,X_v} | \mathcal{F}_t] + bD_{tv} D_{vv_1}, \quad (65)$$

where we have used the fact that D_{vv_1} is deterministic for $t \leq v \leq v_1 \leq T$. By the flow property we have, for $t \leq v \leq v_1 \leq T$,

$$X_{v_1}^{v,X_v} = X_{v_1}^{v,X_v^{t,X_t}} = X_{v_1}^{t,X_t} \quad (66)$$

and by equation (53), or the chain rule, we have

$$D_{tv} D_{vv_1} = e^{-\beta(v-t)} e^{-\beta(v_1-v)} = e^{-\beta(v_1-t)} = D_{tv_1}. \quad (67)$$

Then, substitute equations (65) and (67) into equation (64) to find

$$\begin{aligned} E_T[X_s^{t,X_t} D_{ts} | \mathcal{F}_t] &= X_t + \alpha \beta \int_t^s D_{tv} dv - 2\beta \int_t^s E_T[X_v^{t,X_t} D_{tv} | \mathcal{F}_t] dv \\ &\quad - b\sigma^2 \int_t^s \int_v^T D_{tv_1} dv_1 dv - 2c\sigma^2 \int_t^s \int_v^T E_T[X_{v_1}^{t,X_t} D_{tv_1} | \mathcal{F}_t] dv_1 dv. \end{aligned} \quad (68)$$

For $t \leq v \leq T$ define $\tilde{g}(t, v, x) = E_T[X_v^{t,x} D_{tv}]$. Then, by the Markov property of X_t , $\tilde{g}(t, v, X_t) = E_T[X_v^{t,X_t} D_{tv} | \mathcal{F}_t]$ and, by equation (68), we have that

$$\tilde{g}(t, s, X_t) = X_t + \alpha \beta \int_t^s e^{-\beta(v-t)} dv - 2\beta \int_t^s \tilde{g}(t, v, X_t) dv - b\sigma^2 \int_t^s \int_v^T e^{-\beta(v_1-t)} dv_1 dv - 2c\sigma^2 \int_t^s \int_v^T \tilde{g}(t, v_1, X_t) dv_1 dv. \quad (69)$$

Equation (69) may be solved by considering, for $x \in \mathbb{R}$, the nonlinear integral equation

$$g(t, s, x) = x + \alpha \beta \int_t^s e^{-\beta(v-t)} dv - 2\beta \int_t^s g(t, v, x) dv - b\sigma^2 \int_t^s \int_v^T e^{-\beta(v_1-t)} dv_1 dv - 2c\sigma^2 \int_t^s \int_v^T g(t, v_1, x) dv_1 dv. \quad (70)$$

Differentiating equation (70) with twice respect to s , we obtain the following equivalent ODE

$$g''(t, s, x) = -\alpha \beta^2 e^{-\beta(s-t)} - 2\beta g'(t, s, x) + b\sigma^2 e^{-\beta(s-t)} + 2c\sigma^2 g(t, s, x) \quad (71)$$

with boundary conditions

$$g(t, t, x) = x \quad (72)$$

$$g'(t, t, x) = \alpha \beta - 2\beta x - b\sigma^2 \int_t^T e^{-\beta(v_1-t)} dv_1 - 2c\sigma^2 \int_t^T g(t, v_1, x) dv_1. \quad (73)$$

The ODE (71) has general solution

$$g(t, s, x) = c_1(x) e^{(-\beta + \sqrt{\beta^2 + 2c\sigma^2})s} + c_2(x) e^{(-\beta - \sqrt{\beta^2 + 2c\sigma^2})s} + \frac{b\sigma^2 - \alpha\beta^2}{-\beta^2 - 2c\sigma^2} \cdot e^{-\beta(s-t)}$$

for functions $c_1(x)$ and $c_2(x)$ depending only on x . Applying the boundary conditions (72)-(73) we find that

$$c_1(x) = \frac{\alpha\beta + \frac{\sigma^2 b \eta^2 - 2c\sigma^2(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2} \cdot (e^{-\beta(T-t)} - 1) + x(-\beta + \eta)e^{(-\beta-\eta)(T-t)} + \frac{(b\sigma^2 - \alpha\beta^2)\beta}{\eta^2} + \frac{(b\sigma^2 - \alpha\beta^2)(-\beta + \eta)}{\eta^2} \cdot e^{(-\beta-\eta)(T-t)}}{e^{(-\beta+\eta)T} [\beta + \eta + (-\beta + \eta)e^{2\eta(t-T)}]},$$

$$c_2(x) = \frac{\alpha\beta + \frac{\sigma^2 b \eta^2 - 2c\sigma^2(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2} \cdot (e^{-\beta(T-t)} - 1) + x(-\beta - \eta)e^{(-\beta+\eta)(T-t)} + \frac{(b\sigma^2 - \alpha\beta^2)\beta}{\eta^2} + \frac{(b\sigma^2 - \alpha\beta^2)(-\beta - \eta)}{\eta^2} \cdot e^{(-\beta+\eta)(T-t)}}{e^{(-\beta-\eta)T} [\beta - \eta + (-\beta - \eta)e^{2\eta(T-t)}]},$$

where $\eta = \sqrt{\beta^2 + 2c\sigma^2}$.

Therefore, since $\tilde{g}(t, s, X_t)$ satisfies equation (69) at $x = X_t$ we have by uniqueness of the boundary value problem (71)-(73) that

$$E_T[X_s^{t, X_t} D_{ts} | \mathcal{F}_t] = g(t, s, X_t) \quad (74)$$

for $0 \leq t \leq s \leq T$. That is, $E_T[X_s^{t, X_t} D_{ts} | \mathcal{F}_t]$ is a deterministic function of X_t .

Consider the conditional expectation in equation (59). By Fubini's Theorem and equation (74) we have

$$\begin{aligned} E_T \left[L(t, T, X_t) \middle| \mathcal{F}_t \right] &= - \int_t^T E_T \left[(2cX_u^{t, X_t} + b) D_{tu} \middle| \mathcal{F}_t \right] du = -b \int_t^T D_{tu} du - 2c \int_t^T g(t, u, X_t) du \\ &= \frac{c_1(X_t)(-\beta - \eta)}{\sigma^2} \cdot (e^{(-\beta+\eta)T} - e^{(-\beta+\eta)t}) + \frac{c_2(X_t)(-\beta + \eta)}{\sigma^2} \cdot (e^{(-\beta-\eta)T} - e^{(-\beta-\eta)t}) + \frac{b\eta^2 - 2c(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2} \cdot (e^{-\beta(T-t)} - 1). \end{aligned} \quad (75)$$

Substitute (75) in (59) and write

$$\left. \frac{\partial P(t, T, x)}{\partial x} \right|_{x=X_t} = P(t, T, X_t) (A(\tau)X_t + B(\tau)) \quad (76)$$

where

$$A(\tau) = \frac{2c(e^{2\eta\tau} - 1)}{\beta - \eta + (-\beta - \eta)e^{2\eta\tau}}, \quad (77)$$

$$B(\tau) = \frac{\left[\alpha\beta + \frac{(b\sigma^2 - \alpha\beta^2)\beta}{\eta^2} \right] [(-\beta - \eta)(e^{2\eta\tau} - 1) - 2\eta] + \frac{(b\sigma^2 - \alpha\beta^2)(\beta^2 - \eta^2)}{\eta^2} \cdot (e^{2\eta\tau} - 1) + 2\eta e^{\eta\tau} \cdot \frac{b\sigma^2\eta^2 - 2c\sigma^2(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2}}{\sigma^2 [(\beta + \eta)(e^{2\eta\tau} - 1) + 2\eta]} \quad (78)$$

where

$$\tau = T - t.$$

The ODE (76) has solution

$$P(t, T, X_t) = \exp \left\{ \frac{1}{2} \cdot A(\tau)X_t^2 + B(\tau)X_t + C(T, t) \right\}, \quad (79)$$

where $C(T, t)$ is a differentiable function from \mathbf{R}^2 to \mathbf{R} . By Feynman-Kac Theorem Karatzas and Shreve (1991), $P(t, T, x)$ defined in (54) satisfies the Cauchy problem

$$0 = \frac{\partial P(t, T, x)}{\partial t} + \beta(\alpha - x) \frac{\partial P(t, T, x)}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 P(t, T, x)}{(\partial x)^2} - (cx^2 + bx + a)P(t, T, x), \quad (80)$$

$$1 = P(T, T, x). \quad (81)$$

Substituting (79) in PDE (80) and dividing by $P(t, T, X_t)$, we have

$$\frac{1}{2} X_t^2 \frac{\partial A(\tau)}{\partial t} + X_t \cdot \frac{\partial B(\tau)}{\partial t} + \frac{\partial C(T, t)}{\partial t} + (\beta\alpha - \beta X_t) [A(\tau)X_t + B(\tau)] + \frac{1}{2} \sigma^2 [(A(\tau)X_t + B(\tau))^2 + A(\tau)] = cX_t^2 + bX_t + a \quad (82)$$

Comparing the coefficients on both sides of equation (82), obtain an ODE for $C(T, t)$

$$0 = \frac{\partial C(T, t)}{\partial t} + \beta \alpha B(\tau) + \frac{1}{2} \sigma^2 B(\tau)^2 + \frac{1}{2} \sigma^2 A(\tau) - a \quad (83)$$

$$0 = C(T, T). \quad (84)$$

Solving the ODE (83)-(84), and denoting $C(\tau) = C(T, t)$, we have

$$\begin{aligned} C(\tau) = & \left[\frac{b^2 \sigma^2 - 2\alpha \beta^2 b - 2\alpha^2 \beta^2 c}{2\eta^2} \right] \tau + \frac{1}{2} \log \left\{ \frac{2\eta e^{(\eta+\beta)\tau}}{(\beta+\eta)e^{2\eta\tau} + \eta - \beta} \right\} \\ & + \frac{2c(b\sigma^2 - \alpha\beta^2)^2 + 2\beta(b+2c\alpha)(b\sigma^2 - \alpha\beta^2)(\beta+\eta)e^{\eta\tau} - \beta^2\sigma^2(b+2c\alpha)^2}{\eta^3(\beta+\eta)[(\beta+\eta)e^{2\eta\tau} + \eta - \beta]} \\ & + \frac{\beta^2\sigma^2(b+2c\alpha)^2 - 2c(b\sigma^2 - \alpha\beta^2)^2 - 2\beta(b+2c\alpha)(b\sigma^2 - \alpha\beta^2)(\beta+\eta)}{2\eta^4(\beta+\eta)} - a\tau. \end{aligned} \quad (85)$$

Summarizing the material of this section we have the following result which shows that $P(t, T, X_t)$ is an exponential quadratic function of the factor process.

Theorem 5.1 For $t \in [0, T]$ and for all $x \in \mathbf{R}$,

$$P(t, T, X_t) = \exp \left(\frac{1}{2} \cdot A(\tau) X_t^2 + B(\tau) X_t + C(\tau) \right),$$

where $A(\tau)$, $B(\tau)$ and $C(\tau)$ is given by (77), (78) and (85), respectively.

The Markov property, $P(t, T) = P(t, T, X_t)$ gives the following characterization of the zero coupon bond price.

Corollary 5.2 If the factor process is given by (49) and the short rate is represented by the function (50), the zero-coupon bond price is

$$P(t, T) = \exp \left\{ \frac{1}{2} \cdot A(\tau) X_t^2 + B(\tau) X_t + C(T, t) \right\},$$

where $A(\tau)$, $B(\tau)$ and $C(\tau)$ is given by (77), (78) and (85), respectively.

Corollary 5.2 agrees with the results in Nawalkha et al. (2007, pp. 487-488).

In higher dimensional cases the stochastic flow method requires the addition of some parametric restrictions to the model similar to the ATSM case which was studied by Grasselli and Tebaldi (2007). Nevertheless, the example considered in this section illustrates the similarities of the stochastic flow method in the one-dimensional QTSM with Gaussian factor process to the one-dimensional ATSM model and provides further motivation for the necessity of the change of measure.

6 Conclusions

In this paper we have extended the FBSDE approach, introduced in Hyndman (2009) in the context of affine term structure models, to quadratic term structure models (QTSM) where the factor process is Gaussian and the riskless interest rate is a quadratic functional of the factor process. After characterizing the factor process and the bond price in terms of a coupled quadratic FBSDE under the forward measure we prove the existence and uniqueness of the FBSDE and provide an explicit solution. This approach provides new examples of quadratic FBSDEs with explicit solutions. We extend the FBSDE approach to consider the futures price and forward price of a risky asset with spot price given by an exponential quadratic functional of the factor process, which we term a quadratic price model. Our results are motivated, as in the ATSM case, by the stochastic flows approach of Elliott and van der Hoek (2001), and we briefly consider the one-dimensional QTSM in order to illustrate why the change of measure technique is necessary even though the factors are Gaussian. The results of this paper can be easily extended to consider a Gaussian factor model with time dependent coefficients.

A Appendix

We consider the solvability of a non-symmetric Riccati-type matrix differential equation which includes as special cases those necessary for the solvability of the FBSDEs given in Theorems 3.3, 4.3, and 4.7, as well as their corollaries, which characterize the bond, futures, and forward prices. While the result presented in this appendix is more general than is required for our purposes we believe it is of independent interest as an example of a solvable non-symmetric Riccati-type equation.

The method of proof involves decomposing the non-symmetric Riccati-type differential equation into the sum of the classical symmetric matrix Riccati equation of linear quadratic optimal control (LQC), which is employed by Gombani and Runggaldier (2013), and a skew-symmetric matrix. For further details on the Riccati equations of LQC see, for example, Anderson and Moore (1971) and Barnett (1971). Comprehensive information on Riccati equations can be found in Lancaster and Rodman (1995) and Abou-Kandil et al. (2003).

We relate our results to those presented in Gombani and Runggaldier (2013) for the bond price in a QTSM. However, our approach and model parameterization is different from Gombani and Runggaldier (2013) so we obtain slightly different results. Nevertheless, there is a strong relationship as the next result shows.

Theorem A.1 Consider the general $(n \times n)$ -matrix Riccati-type differential equation for $R_2(t)$

$$\frac{dR_2(t)}{dt} + ([R_2(t)]' + R_2(t))A + \frac{1}{2}([R_2(t)]' + R_2(t))\sigma\sigma'([R_2(t)]' + R_2(t)) - Y = 0 \quad (86)$$

$$R_2(T) = \Theta \quad (87)$$

where $(Y + Y')$ is positive semidefinite and $(\Theta + \Theta')$ is negative semidefinite. Then

(i) a solution $R_2(t)$ to (86)-(87) always exists on the entire interval $[0, T]$ and can be expressed as

$$R_2(t) = -(U(t) + V(t)) \quad (88)$$

where $U(t)$ satisfies

$$\frac{d}{dt}U(t) + U(t)A + A'U(t) - 2U(t)\sigma\sigma'U(t) + Q = 0 \quad (89)$$

$$U(T) = C_1 \quad (90)$$

with $Q = \frac{1}{2}(Y + Y')$ and $C_1 = -\frac{1}{2}(\Theta + \Theta')$ and $V(t)$ is given by

$$V(t) = \tilde{C}_1 + \int_t^T [U(s)A - A'U(s) + \tilde{Q}] ds \quad (91)$$

where $\tilde{Q} = \frac{1}{2}(Y - Y')$ and $\tilde{C}_1 = -\frac{1}{2}(\Theta - \Theta')$.

(ii) A solution to (89)-(90) always exists in the interval $[0, T]$ and it can be expressed as $U(t) = Y(t)X(t)^{-1}$ where X and Y satisfy the linear differential equation

$$\frac{d}{dt} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} A & -2\sigma\sigma' \\ -Q & -A' \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \quad \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = \begin{bmatrix} I \\ C_1 \end{bmatrix}. \quad (92)$$

Moreover, if $\Phi(t, s)$ denotes the fundamental solution (transition matrix) associated with

$$\frac{d}{dt}x(t) = [A - 2\sigma\sigma'U(t)]x(t) \quad (93)$$

then X and Y admit the following interpretation

$$X(t) = \Phi(t, T)$$

and

$$Y(t) = U(t)\Phi(t, T).$$

Proof: The proof of (ii) follows immediately as a special case of Gombani and Runggaldier (2013, Theorem B.1). To prove (i) first suppose $R_2(t)$ is a solution to (86)-(87) on the interval $[0, T]$ and define the symmetric matrix

$$U(t) = -\frac{1}{2}(R_2(t) + [R_2(t)]'). \quad (94)$$

Differentiating equation (94) with respect to t we have, by (86), that $U(t)$ satisfies

$$\begin{aligned} \frac{d}{dt}U(t) &= -\frac{1}{2} \left(\frac{d}{dt}R_2(t) + \frac{d}{dt}[R_2(t)]' \right) \\ &= \frac{1}{2} \left\{ ([R_2(t)]' + R_2(t))A + \frac{1}{2}([R_2(t)]' + R_2(t))\sigma\sigma'([R_2(t)]' + R_2(t)) - \Upsilon \right\} \\ &\quad + \frac{1}{2} \left\{ A'([R_2(t)]' + R_2(t)) + \frac{1}{2}([R_2(t)]' + R_2(t))\sigma\sigma'([R_2(t)]' + R_2(t)) - \Upsilon' \right\} \\ &= -U(t)A - A'U(t) + 2U(t)\sigma\sigma'U(t) - \frac{1}{2}(\Upsilon + \Upsilon') \end{aligned}$$

which gives (89). Evaluating (94) at $t = T$ and applying (87) gives

$$U(T) = -\frac{1}{2}(R_2(T) + [R_2(T)]') = -\frac{1}{2}[\Theta + \Theta'] = C_1$$

which is (90). Next, define the skew-symmetric matrix $V(t)$ by

$$V(t) = -\frac{1}{2}(R_2(t) - [R_2(t)]') \quad (95)$$

and differentiate with respect to t to find that $V(t)$ satisfies

$$\begin{aligned} \frac{d}{dt}V(t) &= -\frac{1}{2} \left(\frac{d}{dt}R_2(t) - \frac{d}{dt}[R_2(t)]' \right) \\ &= -\frac{1}{2} \left\{ -([R_2(t)]' + R_2(t))A - \frac{1}{2}([R_2(t)]' + R_2(t))\sigma\sigma'([R_2(t)]' + R_2(t)) + \Upsilon \right. \\ &\quad \left. + A'([R_2(t)]' + R_2(t)) + \frac{1}{2}([R_2(t)]' + R_2(t))\sigma\sigma'([R_2(t)]' + R_2(t)) - \Upsilon' \right\} \\ &= -\{U(t)A - A'U(t) + \tilde{Q}\}. \end{aligned} \quad (96)$$

Evaluate (95) at $t = T$ and apply (87) to find

$$V(T) = -\frac{1}{2}(R_2(T) - [R_2(T)]') = -\frac{1}{2}(\Theta - \Theta') = \tilde{C}_1. \quad (97)$$

Therefore, solving (96)-(97) gives that $V(t)$ satisfies (91).

Conversely, suppose that $R_2(t)$ is defined by (88)-(91). Note that any solution to (89)-(90) is symmetric and $V(t)$ given by (91) is skew-symmetric. Therefore, by the uniqueness of the decomposition of a square matrix into the sum of symmetric and skew-symmetric matrices, we must have $U(t) = -\frac{1}{2}(R_2(t) + [R_2(t)]')$ and $V(t) = -\frac{1}{2}(R_2(t) - [R_2(t)]')$. Then $R_2(t)$ satisfies

$$\begin{aligned} \frac{d}{dt}R_2(t) &= -\frac{d}{dt}U(t) - \frac{d}{dt}V(t) = -[-U(t)A - A'U(t) + 2U(t)\sigma\sigma'U(t) - Q] - [-U(t)A + A'U(t) - \tilde{Q}] \\ &= 2U(t)A - 2U(t)\sigma\sigma'U(t) + Q + \tilde{Q} \\ &= 2 \left[-\frac{1}{2}(R_2(t) + [R_2(t)]') \right] A - 2 \left[-\frac{1}{2}(R_2(t) + [R_2(t)]') \right] \sigma\sigma' \left[-\frac{1}{2}(R_2(t) + [R_2(t)]') \right] + Q + \tilde{Q} \\ &= -(R_2(t) + [R_2(t)]')A - \frac{1}{2}(R_2(t) + [R_2(t)]')\sigma\sigma'(R_2(t) + [R_2(t)]') + \Upsilon \end{aligned}$$

which is (86) as desired. Finally, evaluating (88) at $t = T$ gives

$$R_2(T) = -(U(T) + V(T)) = -C_1 - \tilde{C}_1 = \frac{1}{2}(\Theta + \Theta') + \frac{1}{2}(\Theta - \Theta') = \Theta$$

so that (87) is satisfied. ■

Remark A.2 Define the Hamiltonian associated with (92)

$$H = \begin{bmatrix} A & -2\sigma\sigma' \\ -Q & -A' \end{bmatrix}.$$

Since H is constant there is an explicit representation

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} I \\ C_1 \end{bmatrix}.$$

Therefore, by Theorem A.1, $R_2(t)$ has an explicit solution on the interval $[0, T]$.

Remark A.3 In the case of the bond price $\Upsilon = \Gamma$ and $\Theta = 0$. Therefore, by Assumption 2.1, since Γ is positive semidefinite, and implicitly symmetric, we have that $\tilde{Q} = 0$. Further, since $\tilde{C}_1 = 0$ we have that, in the case of the bond price, equation (91) simplifies to

$$V(t) = \int_t^T [U(s)A - A'U(s)] ds.$$

In order to provide an exact correspondence with the results of Gombani and Runggaldier (2013) it seems as though we should have $V(t) = 0$ for all $t \in [0, T]$ which is clearly true in the one-dimensional case where A is a scalar, however, in the multi-dimensional case the result is not obvious. Nevertheless, owing to the different parameterization and methods of this paper and those of Gombani and Runggaldier (2013), the difference should be no more alarming than that between the Riccati equations obtained by substitution into the term-structure PDE in Gombani and Runggaldier (2013, equations (2.4)-(2.6)) and those obtained from the LQC approach in Gombani and Runggaldier (2013, equation (3.12)).

We next consider the solution of the associated differential equation for $R_1(t)$ special cases of which appear in Theorems 3.3, 4.3, and 4.7 and their corollaries for the bond, futures, and forward prices.

Corollary A.4 Let $R_1(t)$ be the solution to

$$0 = \frac{d}{dt}R_1(t) + R_1(t) \{A + \sigma\sigma' ([R_2(t)]' + R_2(t))\} + B' ([R_2(t)]' + R_2(t)) - \Psi \quad (98)$$

$$R_1(T) = \theta. \quad (99)$$

Then $R_1(t)$ can be written as

$$R_1(t) = \left(\theta - \int_t^T [2B'Y(s) + \Psi X(s)] ds \right) [X(t)]^{-1}. \quad (100)$$

Proof: The result may be verified by simple differentiation, however, we provide the construction. Taking the transpose of equation (98) and applying the fact that $U(t) = -\frac{1}{2}(R_2(t) + [R_2(t)]')$ we may write

$$\frac{d}{dt}[R_1(t)]' = (2U(t)\sigma\sigma' - A')[R_1(t)]' + K(t) \quad (101)$$

where $K(t) = (2U(t)B + \Psi')$. Let $\Psi(t)$ be the $(n \times n)$ -matrix whose columns are the vectors which form a fundamental set of solutions to

$$\frac{dp(t)}{dt} = (2U(t)\sigma\sigma' - A')p(t) \quad (102)$$

and note that equation (102) is the adjoint equation corresponding to equation (93). Then, by variation of parameters, the solution to (101) with terminal condition (99) is

$$[R_1(t)]' = \Psi(t) \left([\Psi(T)]^{-1}\theta' - \int_t^T [\Psi(s)]^{-1}K(s) ds \right).$$

Define

$$\Psi(t, s) = \Psi(t) \Psi^{-1}(s)$$

and note that $\Psi(t, s)$ is the transition matrix of (102). We also have that, since $\Phi(t, s)$ is the transition matrix of (93) and (102) is the adjoint, that

$$\Psi(t, s) = \left([\Phi(t, s)]^{-1} \right)' = [\Phi(s, t)]'.$$

Therefore, from (101), we may write

$$\begin{aligned} R_1(t) &= \theta[\Psi(t, T)]' - \int_t^T [K(s)]' [\Psi(t, s)]' ds \\ &= \theta\Phi(T, t) - \int_t^T [2B'(s)U(s) + \Psi(s, t)]' \Phi(s, t) ds. \end{aligned} \quad (103)$$

From Theorem A.1 we have $U(s) = Y(s)[X(s)]^{-1}$, $\Phi(s, t) = X(s)[X(t)]^{-1}$, and $X(T) = I$ so the result follows from equation (103). ■

Remark A.5 Since, as noted in Remark A.2, there is an explicit representation for $X(t)$ and $Y(t)$ the integral (100) can be computed explicitly.

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